

On Proving of Diophantine Inequalities with Prime Numbers by Evaluations of the Difference between Consecutive Primes

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Abstract

Using as the working hypothesis of an evaluation of the difference between primes $p_{n+1} - p_n = O(\sqrt{p_n})$ we represent in detail the proofs of Legendre's and Oppermann's conjectures.

1 Introduction

Applying the best available evaluation of the difference between primes, $p_{n+1} - p_n = O(p_n^{0.525})$ [1] we have obtained proofs for some Diophantine inequalities with primes including Ingham's results [7]. Some authors believe that in the presence of a stronger evaluations of the difference between consecutive primes it may be possible to prove Legendre's conjecture and some other statements [6]. The generally expected evaluation of the difference between consecutive primes is $p_{n+1} - p_n = O(\sqrt{p_n})$ [3], [5]. In this paper, using $p_{n+1} - p_n = O(\sqrt{p_n})$ we are able to prove Legendre's and Oppermann's conjectures.

2 $p_{n+1} - p_n = O(\sqrt{p_n})$ and Diophantine inequalities with primes

Proposition 2.1. *The interval $(n, n + \lambda \sqrt{n})$ contains a prime for every integer $n \geq c(\lambda)$ where $\lambda, c(\lambda)$ are some constants, if and only if $p_{k+1} - p_k < \lambda \sqrt{p_k}$ is true for all primes $p_k \geq c(\lambda)$.*

Proof. Let $(n, n + \lambda \sqrt{n})$ contain primes for every integer $n \geq c(\lambda)$. Then for $n = p_k$ the interval $(p_k, p_k + \lambda \sqrt{p_k})$ contains a prime q . Hence we have $p_k < q < p_k + \lambda \sqrt{p_k}$. Since $p_k < p_{k+1} \leq q$, $p_{k+1} - p_k < \lambda \sqrt{p_k}$ is true.

Let $p_{k+1} - p_k < \lambda \sqrt{p_k}$ be true for every prime $p_k \geq c(\lambda)$. Let n_0 be such that $(n_0, n_0 + \lambda \sqrt{n_0})$ contains no primes. Let p_{n-1}, p_n be such that $p_{n-1} < n_0 < p_n$. Then $(p_{n-1}, p_{n-1} + \lambda \sqrt{p_{n-1}})$ contains no primes. Since n_0 is not prime, the interval $(p_{n-1}, n_0 + \lambda \sqrt{n_0}) = (p_{n-1}, n_0) \cup [n_0] \cup (n_0, n_0 + \lambda \sqrt{n_0})$ contains no primes. Furthermore, $(p_{n-1}, p_{n-1} + \lambda \sqrt{p_{n-1}}) \subset (p_{n-1}, n_0 + \lambda \sqrt{n_0})$ since $p_{n-1} + \lambda \sqrt{p_{n-1}} < n_0 + \lambda \sqrt{n_0}$, so $(p_{n-1}, p_{n-1} + \lambda \sqrt{p_{n-1}})$ contains no primes, contradicting $p_{k+1} - p_k < \lambda \sqrt{p_k}$. \square

Corollary 2.2. Let $\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{\lambda}{(\sqrt{1.2}+1)}$, where $\lambda, c(\lambda) > 25$ are constants, be true for all primes $p_n \geq c(\lambda)$. Then $p_{n+1} - p_n < \lambda \sqrt{p_n}$ is true for every $p_n \geq c(\lambda)$.

Proof. Since according to [4] for any pair of neighbouring primes, $p_{n+1} < \frac{6}{5}p_n$, where $p_n > 25$ is true; $p_{n+1} - p_n < \lambda \sqrt{p_n}$ is also true for every prime $p_n \geq c(\lambda)$. \square

Corollary 2.3. Let $\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{\lambda}{(\sqrt{1.2}+1)}$, where $\lambda, c(\lambda) > 25$ are constants, be true for all primes $p_n \geq c(\lambda)$. Then the interval $(n, n + \lambda \sqrt{n})$ contains a prime for every integer $n \geq c(\lambda)$.

Proof. Corollary 2.3 is a consequence of proposition 2.1 and corollary 2.2. \square

Proposition 2.4. The interval $(n, n + g(n) \sqrt{n})$, where $g(n) = o(1)$ and $g(n) \sqrt{n}$ is a non-decreasing function, contains a prime for every integer $n \geq c(g)$, where $c(g)$ is some constant; if and only if $p_{k+1} - p_k < g(p_k) \sqrt{p_k}$ is true for any prime $p_k \geq c(g)$.

Proof. Let $n + g(n) \sqrt{n}$ contain a prime for every integer $n \geq c(g)$. Then $(p_k, p_k + g(p_k) \sqrt{p_k})$, where $n = p_k$, contains a prime q such that $p_k < q < p_k + g(p_k) \sqrt{p_k}$. Since $p_k < p_{k+1} \leq q$, $p_{k+1} - p_k < g(p_k) \sqrt{p_k}$ is true.

Let $p_{k+1} - p_k < g(p_k) \sqrt{p_k}$ be true for every $p_k \geq c(g)$. Let n_0 be such an integer that the interval $(n_0, n_0 + g(n_0) \sqrt{n_0})$ contains no primes. Let p_{n-1}, p_n be such that $p_{n-1} < n_0 < p_n$, hence the interval $(p_{n-1}, p_{n-1} + g(p_{n-1}) \sqrt{p_{n-1}})$ contains no primes. Since n_0 is not prime, the interval $(p_{n-1}, n_0 + g(n_0) \sqrt{n_0}) = (p_{n-1}, n_0) \cup [n_0] \cup (n_0, n_0 + g(n_0) \sqrt{n_0})$ contains no primes. $(p_{n-1}, p_{n-1} + g(p_{n-1}) \sqrt{p_{n-1}}) \subset (n_0, n_0 + g(n_0) \sqrt{n_0})$ since $p_{n-1} + g(p_{n-1}) \sqrt{p_{n-1}} < n_0 + g(n_0) \sqrt{n_0}$; then the interval $(p_{n-1}, p_{n-1} + g(p_{n-1}) \sqrt{p_{n-1}})$ contains no primes, contradicting $p_{k+1} - p_k < g(p_k) \sqrt{p_k}$. \square

Corollary 2.5. Let $g(n) = o(1)$ and there exists a constant $c(g)$ such that the interval $(n, n + g(n) \sqrt{n})$ contains a prime for every integer $n \geq c(g)$, then $\sqrt{p_{m+1}} - \sqrt{p_m} = o(1)$ is true.

Proposition 2.6. $p_{n+1} - p_n = O(f(p_n))$ is true if and only if $\sqrt{p_{n+1}} - \sqrt{p_n} = O(\frac{f(p_n)}{\sqrt{p_n}})$ is true.

Proof. Let $p_{n+1} - p_n = O(f(p_n))$ be true. Then there exist such constants k, N_k that $p_{n+1} - p_n < kf(p_n)$ is true for every $p_n \geq N_k$. Hence, $\sqrt{p_{n+1}} - \sqrt{p_n} = O(\frac{f(p_n)}{\sqrt{p_n}})$ is true.

Let $\sqrt{p_{n+1}} - \sqrt{p_n} = O(\frac{f(p_n)}{\sqrt{p_n}})$ be true. Then there exist such constants $k, N_k > 25$ that $\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{kf(p_n)}{\sqrt{p_n}}$ is true for any $p_n \geq N_k$. Then, $p_{n+1} - p_n < (\sqrt{1.2} + 1)kf(p_n)$ is true according to [4], and therefore $p_{n+1} - p_n = O(f(p_n))$ is also true. \square

Proposition 2.7. Let Cramer's conjecture be true, then there exists some infinite subset of primes E such that for every prime $p_n \in E$, $\frac{\ln(p_n)}{\sqrt{p_n}} < \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{k \log^2(p_n)}{\sqrt{p_n}}$ is true.

Proof. According to Cramer's conjecture [2], $p_{n+1} - p_n = O(\log^2(p_n))$ and proposition 2.6, $\sqrt{p_{n+1}} - \sqrt{p_n} = O(\frac{\log^2(p_n)}{\sqrt{p_n}})$ is true. Then, there exist such k, N_k that for every $p_n \geq N_k$, $\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{k \log^2(p_n)}{\sqrt{p_n}}$ is true. Furthermore, $p_{n+1} - p_n = O(\log(p_n))$ is not true according to E. Westzynthius and so $\sqrt{p_{n+1}} - \sqrt{p_n} = O(\frac{\ln(p_n)}{\sqrt{p_n}})$ is also not true according to proposition 2.6. Therefore there exists an infinite set of primes S such that $\frac{\ln(p_n)}{\sqrt{p_n}} < \sqrt{p_{n+1}} - \sqrt{p_n}$ is true. Taking $E = \{p_n \in S | p_n \geq N_k\}$, the inequality $\frac{\ln(p_n)}{\sqrt{p_n}} < \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{k \log^2(p_n)}{\sqrt{p_n}}$ is true for any $p_n \in E$. \square

3 Legendre's conjecture

Conjecture 3.1 (Legendre). *The interval $(n^2, (n+1)^2)$ contains a prime for any $n \in \mathbb{N}$.*

Lemma 3.2. *The interval $(n-2\sqrt{n}, n)$ contains a prime for all $n \geq 4$ if and only if $p_k - p_{k-1} < 2\sqrt{p_k}$ is true for all $p_k \geq 3$.*

Proof. Let $(n-2\sqrt{n}, n)$ contain a prime, then the interval $(p_k - 2\sqrt{p_k}, p_k)$ where $n = p_k$ contains a prime q . Therefore $p_k - 2\sqrt{p_k} < q < p_k$, and since $q \leq p_{k-1} < p_k$, $p_k - p_{k-1} < 2\sqrt{p_k}$ is true.

Let $p_k - p_{k-1} < 2\sqrt{p_k}$ be true for all $p_k \geq 3$, but there exists such n_0 that $(n_0 - 2\sqrt{n_0}, n_0)$ contains no primes. Let p_{n-1}, p_n be such primes that $p_{n-1} < n_0 < p_n$. Then the interval $(p_n - 2\sqrt{p_n}, p_n)$ contains no primes. Since n_0 is not prime, the interval $(n_0 - 2\sqrt{n_0}, p_n) = (n_0 - 2\sqrt{n_0}, n_0) \cup [n_0] \cup (n_0, p_n)$ contains no primes. Moreover, $(p_n - 2\sqrt{p_n}, p_n) \subset (n_0 - 2\sqrt{n_0}, p_n)$ since $n_0 - 2\sqrt{n_0} < p_n - 2\sqrt{p_n}$ so the interval $(p_n - 2\sqrt{p_n}, p_n)$ contains no primes, contradicting $p_k - p_{k-1} < 2\sqrt{p_k}$. \square

Proof of conjecture 3.1 (Legendre). Let $p_{n+1} - p_n < 2\sqrt{p_{n+1}}$ be true, then according to lemma 3.2 the interval $(m^2 - 2m, m^2)$ where $n = m^2$ contains a prime. Since $(m^2 - 2m, (m-1)^2)$ contains no integers and $(m-1)^2$ is not prime, then $((m-1)^2, m^2)$ contains a prime for every $m \geq 2$. \square

4 Oppermann's conjecture

Conjecture 4.1 (Oppermann). *The interval $(n^2, (n+1)^2)$ contains two primes for any $n \in \mathbb{N}$.*

Proposition 4.2. *The intervals $(l - \sqrt{l}, l)$ and $(l, l + \sqrt{l})$ contain primes for every $l \geq p_{32} = 131$ if and only if $p_k - p_{k-1} < \sqrt{p_{k-1}}$ is true every prime $p_k \geq 131$.*

Proof. Let $(l - \sqrt{l}, l)$ and $(l, l + \sqrt{l})$ contain primes for every prime $l \geq p_{32} = 131$. Let p and q respectively belong to the intervals $(p_n - \sqrt{p_n}, p_n)$, $(p_n, p_n + \sqrt{p_n})$ where $l = p_n$. Since $p \leq p_{n-1} < p_n$ and $p_n < p_{n+1} \leq q$, so p_{n-1} and p_{n+1} also belong to $(p_n - \sqrt{p_n}, p_n)$, $(p_n, p_n + \sqrt{p_n})$. Thus:

$$p_n - p_{n-1} < p_n - (p_n - \sqrt{p_n}) = \sqrt{p_n}, p_{n+1} - p_n < (p_n + \sqrt{p_n}) - p_n = \sqrt{p_n} \quad (1)$$

and since $p_{32} - p_{31} < \sqrt{p_{31}}$, therefore $p_k - p_{k-1} < \sqrt{p_{k-1}}$ is true for every prime $p_k \geq p_{32}$.

Let $p_k - p_{k-1} < \sqrt{p_{k-1}}$ is true for every prime $p_k \geq p_{32}$ and $l = p_n \geq p_{32}$, then $p_n - p_{n-1} < \sqrt{p_{n-1}} < \sqrt{p_n}$ and also $p_{n+1} - p_n < \sqrt{p_n}$ hence $p_{n-1} \in (p_n - \sqrt{p_n}, p_n)$ and $p_{n+1} \in (p_n, p_n + \sqrt{p_n})$. Thus we have that the intervals $(p_n - \sqrt{p_n}, p_n), (p_n, p_n + \sqrt{p_n})$ contain primes. \square

Proposition 4.3. *The intervals $(n - \sqrt{n}, n)$ and $(n, n + \sqrt{n})$ contain primes for every integer $n \geq 131$ if and only if $p_k - p_{k-1} < \sqrt{p_{k-1}}$ is true for all primes $p_k \geq 131$.*

Proof. Let $(n - \sqrt{n}, n)$ and $(n, n + \sqrt{n})$ contain primes for all integers $n \geq 131$ and $n = p_k$. Then according proposition 4.2 $p_k - p_{k-1} < \sqrt{p_{k-1}}$ is true for all $p_k \geq 131$. Let $p_k - p_{k-1} < \sqrt{p_{k-1}}$ be true for all $p_k \geq 131$ but proposition 4.2 is false for some integer $n > p_{32} = 131$. Let n_0 be such an integer that at least one of the intervals $(n_0 - \sqrt{n_0}, n_0), (n_0, n_0 + \sqrt{n_0})$ contains no primes; then there are two cases:

Case 1: Let $(n_0 - \sqrt{n_0}, n_0)$ contain no primes. Let p_{n-1}, p_n be such that $p_{32} \leq p_{n-1} < n_0 < p_n$, then the interval $(p_n - \sqrt{p_n}, p_n)$ contains no primes. Indeed, n_0 is not prime and the interval $(n_0 - \sqrt{n_0}, p_n) = (n_0 - \sqrt{n_0}, n_0) \cup [n_0] \cup (n_0, p_n)$ contains no primes. Further we have $(p_n - \sqrt{p_n}, p_n) \subset (n_0 - \sqrt{n_0}, p_n)$ since $n_0 - \sqrt{n_0} < p_n - \sqrt{p_n}$ the interval $(p_n - \sqrt{p_n}, p_n)$ contains no primes.

Case 2: Let $(n_0, n_0 + \sqrt{n_0})$ contain no primes. Let p_{n-1}, p_n be such that $p_{32} \leq p_{n-1} < n_0 < p_n$ then the interval $(p_{n-1}, p_{n-1} + \sqrt{p_{n-1}})$ contains no primes. Indeed, n_0 is not prime so the interval $(p_{n-1}, n_0 + \sqrt{n_0}) = (p_{n-1}, n_0) \cup [n_0] \cup (n_0, n_0 + \sqrt{n_0})$ contains no primes. Furthermore, $(p_{n-1}, p_{n-1} + \sqrt{p_{n-1}}) \subset (p_{n-1}, n_0 + \sqrt{n_0})$ since $p_{n-1} + \sqrt{p_{n-1}} < n_0 + \sqrt{n_0}$ so the interval $(p_{n-1}, p_{n-1} + \sqrt{p_{n-1}})$ contains no primes.

Both cases contradict proposition 4.2 since $p_{32} < p_n$ in case 1 and $p_{32} \leq p_{n-1}$ in case 2. \square

Proof of conjecture 4.1 (Oppermann). Let $p_k - p_{k-1} < \sqrt{p_{k-1}}$ be true for every $p_k \geq 131$ then according to proposition 4.3 the intervals $(m^2 + m, (m+1)^2)$ where $n = (m+1)^2$ and $(m^2, m^2 + m)$ where $n = m^2, m^2 > 131$ contain primes. The interval $(m^2, (m+1)^2)$ is a union of $(m^2, m^2 + m), (m^2 + m, (m+1)^2)$. Thus the conjecture is true for all m^2 not less than 131; by actual verification we find that it is true for smaller values. \square

5 Discussion & Conclusion

The paper has explicitly shown that the general expected evaluation of the difference between consecutive primes $p_{n+1} - p_n = O(\sqrt{p_n})$ is a sufficient condition to prove Legendre's and Oppermann's conjectures. We have proved Legendre's and Oppermann's conjectures applying as evaluations of the difference between primes $p_{n+1} - p_n < 2\sqrt{p_{n+1}}$ and $p_{n+1} - p_n < \sqrt{p_n}$, respectively.

References

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